increase in strength of colour, which was generally observed in the light transmitted through these films when the plane of polarisation of obliquely incident light was changed from that of incidence to a perpendicular position is accounted for.

In Part III some evidence is brought to show that the allotropic silvers obtained by Carey Lea* are particular cases of the media which have been considered in the second part.

"The General Theory of Integration." By W. H. Young, Sc.D., St. Peter's College, Cambridge. Communicated by Dr. E. W. Hobson, F.R.S. Received April 23,—Read May 19, 1904.

(Abstract.)

The paper begins with a recapitulation of the well-known definitions of integration and of upper and lower integration (intégral par excès, par défaut; oberes, unteres Integral). The theorem on which the Darboux definition of upper (lower) integration is founded is stated and proved in the following form:—

Given any small positive quantity e_1, we can determine a positive quantity e, such that, if the fundamental segment S be divided up in any manner into a finite number of intervals, then, provided only the length of each interval is less than e, the upper summation of any function over these intervals differs by less than e_1 from a definite limiting value (the upper integral).

Next follows a discussion as to whether it is admissible to adopt a more general mode of division of the fundamental segment than that used by Riemann, Darboux and other writers, when forming summations (upper, lower summations), defining as limit the integral (upper, lower integral), of a function over the fundamental segment. It is shown by examples first that the restriction as to the finiteness of the number of intervals into which the fundamental segment is divided cannot be removed without limitations; but that it can be removed, provided the content of the intervals is always equal to that of the fundamental segment. Secondly it is shown that the error introduced by taking the summation over an infinite number of intervals whose content is less than that of the fundamental segment, is not in general corrected by adding to the summation the content of the points external to the intervals multiplied by corresponding value (upper, lower limit) of the function. Similarly it is shown that the more

* 'Amer. Journ. of Science,' 1886.

general division of the fundamental segment into component sets of points, whose content plays the part of the length of the intervals in the original definitions, leads to summations which do not, in general, have a definite limit even for integrable functions. The lower limit of such generalised upper summations is shown to be not less than the upper limit of such generalised lower summations; but it is shown that in general, only in case of upper continuous functions does the former give us the upper integral, and in the case of lower semi-continuous functions does the latter give us the lower integral. In general, introducing the terms *outer and inner measure of the integral* for these limits, the lower integral is less than the inner measure, which is less than the outer measure, which is less than the upper integral.

The property of semi-continuous functions just mentioned leads to a new form of the definition of the upper (lower), integral in this case, namely, as follows:—

> Divide the fundamental segment S into a finite or countably infinite number of measurable components, multiply the content of each component by the upper (lower) limit of the values of an upper (lower) semi-continuous function at points of that component and sum all such products; then the lower (upper) limit of all such summations for every conceivable mode of division is the upper (lower) integral of the semi-continuous function.

Introducing *upper and lower limiting functions*,* we then have the following theorem:—

> The upper (lower) integral of any function is the upper (lower) integral of its associated upper (lower) semi-continuous function.

This leads to a new definition of upper and lower integration, which is as follows:—

> Divide the fundamental segment into any finite or countably infinite number of measurable components, multiply the content of each component by the upper (lower) limit of the maxima (minima) of the function at points of that component and sum all such products; then the lower (upper), limit of all such summations for every conceivable mode of division is the upper (lower) integral of the function over the fundamental segment.

This gives us also a definition of the integral in the case when it exists, that is, when the upper and lower integrals are equal.

This form of the definitions is at once extendable to the case when the fundamental set S is any measurable set whatever, we merely have to replace the word segment by set, or more precisely by measurable

* "On Upper and Lower Integration," 'Lond. Math. Soc. Proc.'

set. A particular form of division of S, analogous to that by means of intervals of the same content as the fundamental segment, is shown to lead infallibly to the upper and lower integrals of any function with respect to S; this mode of division is called *division of* S *by means of segments* (e, e'), it is such that each component lies inside a corresponding interval of length less than e, the content of these intervals being less than $S + e'$, and the points of S which are not internal to the intervals forming a set of zero content.

Based on this division of the fundamental set, we have an alternative definition of upper and lower integration with respect to a fundamental set, which is more nearly allied to the Darboux definitions for the case when the fundamental set is a finite segment. This is as follows:—

> Let the fundamental set, excluding at most a set of points of zero content, be enclosed in or on the borders of a set of non-overlapping segments each less than e, and of content less than $S+e$. Then let the content of that component of S in any segment be multiplied by the upper (lower) limit of the values of the function at points of that component, and let the summation be formed of all such products. Then it may be shown that this summation has a definite limit when e is indefinitely decreased, independent of the mode of construction of the segments and the mode in which e approaches the value zero. This limit is called the upper (lower) integral of the function with respect to the fundamental set S.

In the case when the upper and lower integrals coincide, the function is said to be integrable with respect to S, and the condition of integrability is found in a form agreeing completely with Riemann's condition in the case when S is a segment. To prove this the theorem is required that the *sum of any finite number of upper integrals of upper semi-continuous functions with respect to a fundamental set* S *is the upper integral of their sum*, and the proof of this theorem is given.

It is then shown that, except in the case of upper (lower) semi-continuous functions, the upper (lower) integral over the fundamental set S is not necessarily equal to the sum of the upper (lower) integrals over any set of components of S, but that this is the case when S is divided by means of segments (e, e').

A function which is integrable with respect to S is shown to have the following properties:—

(1) It is integrable over every component set of S.

(2) The integral of the integrable function is equal to the sum of the integrals over every finite or countably infinite number of components into which S may be divided.

(3) The sum of the integrals of any finite number of integrable

functions over S is equal to the integral of the sum of those functions over S.

In § 21 the calculation of upper and lower integrals with respect to any fundamental set S is reduced to a problem of ordinary integration. The formulæ, which are similar in form to those already given by the author for the case when S is a finite segment, in a paper presented to the London Mathematical Society, are as follows:—

The upper integral of any function with respect to a measurable set S is

$$\mathrm{KS} + \int_{\mathrm{K}}^{\mathrm{K}'} \mathrm{I}\,dk,$$

where K is any quantity not greater than the lower limit, and K′ not less than the upper limit of the function for points of S, I being the content of that component of S at every point of which the maximum of the function is greater than or equal to k.

The lower integral is

$$\mathrm{K}'\mathrm{S} - \int_{\mathrm{K}}^{\mathrm{K}'} \mathrm{J}\,dk,$$

J being the content of that component of S at every point of which the minimum of the function is less than or equal to k.

These formulæ lead to certain theorems with respect to the distribution of the values of an ordinary continuous function and of an integrable function.

The remainder of the paper is devoted to the discussion of the inner and outer measures of the integral of any function, and in the case when they are equal of the generalised integral of a function, which is, in this generalised sense, integrable. In particular it is shown that such functions are none other than the functions which Lebesgue has named *summable*, and the generalised integral is shown to be identical with the Lebesgue integral in the case when S is a finite segment; a geometrical interpretation of the integral, similar to that used by Lebesgue, is given in the general case.

Contrasting the first definition given of the generalised integral with the geometrical definition, it is seen that they stand to one another in the same relation as the ordinary definition of integration, say of a continuous function, to its definition as a certain area. Just as, however, the mathematical concept of area is more complex than, and, indeed, depends on that of length, so does the theory of the content of a plane set of points depend naturally on that of a linear set. Just as the determination of area requires the application of the processes explained in the first definition of integration of continuous functions, so with the content of a plane set. Thus the comparative simplicity of the geometrical definition is only apparent.

Lebesgue's theorem that the sum of two summable functions is a summable function and its integral is the sum of their integrals is then proved by geometrical considerations, and a more general theorem is given, viz.:—

> If X^0 and X^1 be the outer and inner measures of the content of the ordinate section of a measurable set by the ordinate through the point x, X^0 and X^1 are both summable functions, and the generalised integral of either is the content of the measurable set.

It is here assumed that the content of the set got by closing the measurable set is finite. The content of any measurable set, with this restriction only, is thus obtained in the form of a generalised integral and, therefore, of an ordinary integral; in fact—

> The content of any measurable set (provided the set got by closing it has finite content) is $\int I dx$.

Here I is the content of the component of the fundamental set at which the inner (or the outer) measure of the content of the ordinate section of the given set is greater or equal to k.

It is to be remarked that though in this abstract reference has only been made to linear and plane sets and to the corresponding integrals, the arguments are perfectly general and apply to space of any number of dimensions. For instance the concluding result is as follows:—

> To find the content of a measurable n-dimensional set, take any hyperplane section and project the whole set on to this hyperplane. Any measurable set containing this projection we take as the fundamental set S. Divide S up in any way into a finite or countably infinite set of measurable components, and multiply the content of each component by the upper (lower) limit of the values of the (linear) inner or outer content of the corresponding ordinate sections of the given set, summing all such products, the lower (upper) limit of all such summations is the content of the given set.